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AnalyticBridge Mathematical Competition

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Preview

What do you do on vacation? I'd like to say that I rest, or that I visit interesting places, or that I have fun by the sea. I promise I'll do all of it during my current holidays, but I started on the wrong foot: I worked on a mathematical problem proposed by Vincent Granville.

The Problem

Let me state the problem here. In the following, x is any permutation of the integers from *0* to *n-1*. Let's define three functions

$u(x) = sum_i |x(i) - i|$

$v(x) = sum_i |(x(i) - (n-1- i)|$

$t(x) = min(u(x), v(x))$

where $|a|$ is the absolute value of *a*.

The problem is to compute $q(n)$ defined as the maximum of $t(x)$ over all permutations x of the integers *0..n-1*.

For reasons that will be clearer later, let *m* be the integral qutotient of the division of n by 4, and *r* be the remainder, i.e.

$n = 4m + r$

$0 <= r < 4$

An Upper Bound

The key idea to compute a good upper bound on $q(n)$ is to remark that $t(x) <= 1/2\ w(x)$ where $w(x) = u(x) + v(x)$, and then compute the maximum w_n of $w(x)$ over all permutations *x* of the integers *0..n-1*. We will then have

$q(n) <= 1/2\ w_n$

We have $w(x) = sum\_i\ \ f(i,x_i)$ where $f(i,j) = |j - i| + |j - (n-1-i)|$

Let's study the function *f*. Its value is given by

$f(i,j) = 2j - (n-1)$ if $j >= i$ & $j >= n-1-i$

$f(i,j) = 2i - (n-1)$ if $j <= i$ & $j >= n-1-i$

$f(i,j) = -2i + (n-1)$ if $j >= i$ & $j <= n-1-i$

$f(i,j) = -2j + (n-1)$ if $j <= i$ & $j <= n-1-i$

Let's compute the value $f(i,j)$ for each cell (i,j) of the *n* by *n* square. It is easy to check that the cells having the same value are arranged in concentric rings as depicted in Fig. 1.

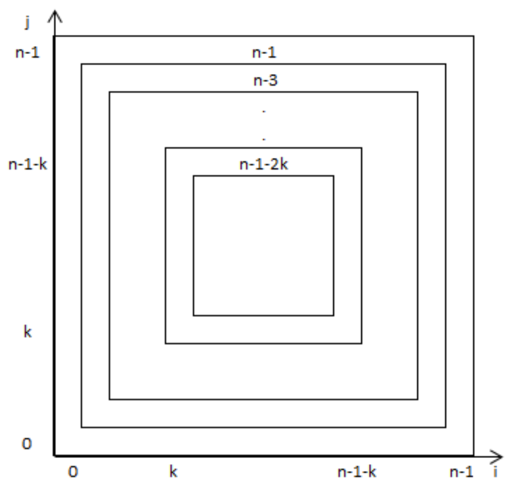

Figure 1. Value of *f* for cells of the nxn square

The value of cells in ring R_k is $n-1-2k$ where R_k is the set of cells (i,j) such that one of the following conditions is true

(i) $k <= i <= n-1-k$ & $j = k$

(ii) $k <= i <= n-1-k$ & $j = n-1-k$

(ii) $k <= j <= n-1-k$ & $i = k$

(iv) $k <= j <= n-1-k$ & $i = n-1-k$

A permutation *x* is equivalent to a set of *n* cells in the square such that there is exactly one cell per row, and exactly one cell per column. This set is $\{ (i,x(i)) \mid 0 <= i < n\}$. Note that there are at most 4 such cells in a given ring (at most one for each side of the ring).

Finding the maximum of w(x) over all permutations amounts to selecting such set that maximizes the sum of the value of its cells. Given the value of cells in rings decreases with k, the maximum is obtained by selecting as many cells as possible from the rings in increasing values of *k*. It means selecting 4 cells in each of the first m rings then r cells in the m-th ring, where m is quotient of n/4 and r is the remainder. The corresponding value of w(x) is

$$w_n = sum_{0<=k<m}(n-1-k) + r(n-1-m)$$

Elementary calculus gives

$$w_n = 12\ m^2 + 6\ mr + r(r-1)$$

Therefore, we have

$$(1) \qquad q(n) <= 6\ m^2 + 3\ mr + r(r-1)/2$$

A Lower Bound

Let's now look at lower bounds for *q(n)*. It will depend on the value of *r*. We will simply use that *q(n) >= t(x)* where *x* is a permutation.

Case r = 0

We have $n = 4m$. Let's define the permutation x^0 as follows.

x0(i) = 2m+i	if *0 <= i < m*
x0(i) = 2m-1-i	if *m <= i < 2m*
x0(i) = 6m-1-i	if *2m <= i < 3m*
x0(i) = -2m+i	if *3m <= i < n*

It is depicted in Fig2. It is easy to check the following

$$t(x^0) = 6\ m^2$$

Therefore we have

$$(2) \qquad q(4m) >= 6m^2$$

From (1) above and (2) we get

$$(3) \qquad q(4m) = 6m^2$$

Equivalently

$$(4) \qquad q(n) = 3/8\ n^2 \text{ if } n \text{ is a multiple of 4}$$

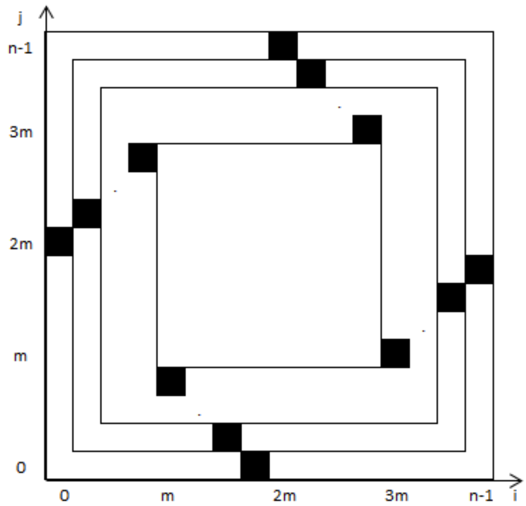

Figure 2. A permutation

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