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What do you do on vacation? I'd like to say that I rest, or that I visit interesting places, or that I have fun by the sea. I promise I'll do all of it during my current holidays, but I started on the wrong foot: I worked on a [mathematical problem](#) proposed by Vincent Granville.

The Problem

Let me state the problem here. In the following, x is any permutation of the integers from 0 to $n-1$. Let's define three functions

$$u(x) = \sum_i |x(i) - i|$$

$$v(x) = \sum_i |x(i) - (n-1-i)|$$

$$t(x) = \min(u(x), v(x))$$

where $|a|$ is the absolute value of a .

The problem is to compute $q(n)$ defined as the maximum of $t(x)$ over all permutations x of the integers $0..n-1$.

For reasons that will be clearer later, let m be the integral quotient of the division of n by 4 , and r be the remainder, i.e.

$$n = 4m + r$$

$$0 \leq r < 4$$

An Upper Bound

The key idea to compute a good upper bound on $q(n)$ is to remark that $t(x) \leq 1/2 w(x)$ where $w(x) = u(x) + v(x)$, and then compute the maximum w_n of $w(x)$ over all permutations x of the integers $0..n-1$. We will then have

$$q(n) \leq 1/2 w_n$$

We have $w(x) = \sum_i f(i, x_i)$ where $f(i, j) = |j - i| + |j - (n-1-i)|$

Let's study the function f . Its value is given by

$$f(i, j) = 2j - (n-1) \quad \text{if } j \geq i \text{ \& } j \geq n-1-i$$

$$f(i, j) = 2i - (n-1) \quad \text{if } j \leq i \text{ \& } j \geq n-1-i$$

$$f(i, j) = -2i + (n-1) \quad \text{if } j \geq i \text{ \& } j \leq n-1-i$$

$$f(i, j) = -2j + (n-1) \quad \text{if } j \leq i \text{ \& } j \leq n-1-i$$

Let's compute the value $f(i, j)$ for each cell (i, j) of the n by n square. It is easy to check that the cells having the same value are arranged in concentric rings as depicted in Fig. 1.

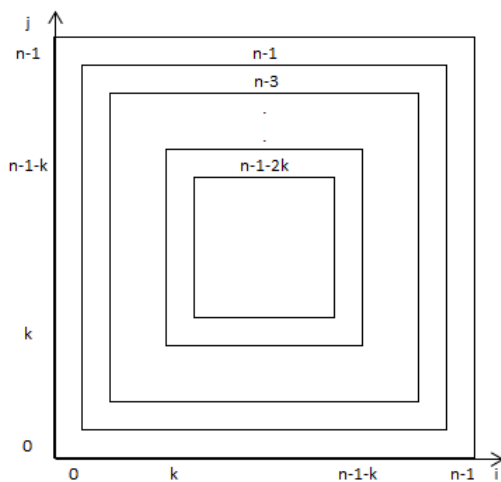


Figure 1. Value of f for cells of the $n \times n$ square

The value of cells in ring R_k is $n-1-2k$ where R_k is the set of cells (i, j) such that one of the following conditions is true

- (i) $k \leq i \leq n-1-k \text{ \& } j = k$
- (ii) $k \leq i \leq n-1-k \text{ \& } j = n-1-k$
- (ii) $k \leq j \leq n-1-k \text{ \& } i = k$
- (iv) $k \leq j \leq n-1-k \text{ \& } i = n-1-k$

A permutation x is equivalent to a set of n cells in the square such that there is exactly one cell per row, and exactly one cell per column. This set is $\{(i, x(i)) \mid 0 \leq i < n\}$. Note

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that there are at most 4 such cells in a given ring (at most one for each side of the ring).

Finding the maximum of $w(x)$ over all permutations amounts to selecting such set that maximizes the sum of the value of its cells. Given the value of cells in rings decreases with k , the maximum is obtained by selecting as many cells as possible from the rings in increasing values of k . It means selecting 4 cells in each of the first m rings then r cells in the m -th ring, where m is quotient of $n/4$ and r is the remainder. The corresponding value of $w(x)$ is

$$w_n = \sum_{0 < k < m} (n-1-k) + r(n-1-m)$$

Elementary calculus gives

$$w_n = 12 m^2 + 6 m r + r(r-1)$$

Therefore, we have

$$(1) \quad q(n) \leq 6 m^2 + 3 m r + r(r-1)/2$$

A Lower Bound

Let's now look at lower bounds for $q(n)$. It will depend on the value of r . We will simply use that $q(n) \geq t(x)$ where x is a permutation.

Case $r = 0$

We have $n = 4m$. Let's define the permutation x^0 as follows.

$$\begin{aligned} x^0(i) &= 2m+i && \text{if } 0 \leq i < m \\ x^0(i) &= 2m-1-i && \text{if } m \leq i < 2m \\ x^0(i) &= 6m-1-i && \text{if } 2m \leq i < 3m \\ x^0(i) &= -2m+i && \text{if } 3m \leq i < n \end{aligned}$$

It is depicted in Fig2. It is easy to check the following

$$t(x^0) = 6 m^2$$

Therefore we have

$$(2) \quad q(4m) \geq 6m^2$$

From (1) above and (2) we get

$$(3) \quad q(4m) = 6m^2$$

Equivalently

$$(4) \quad q(n) = 3/8 n^2 \text{ if } n \text{ is a multiple of } 4$$

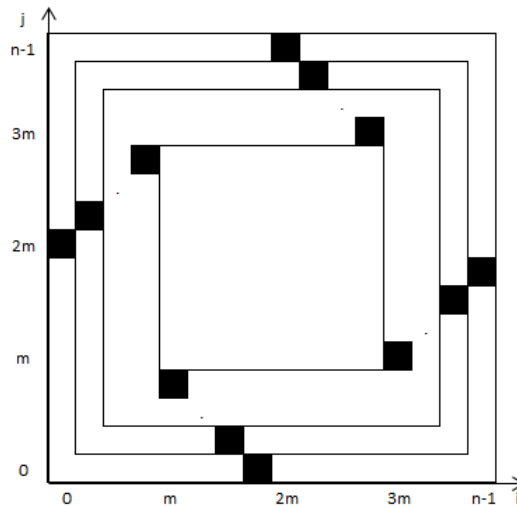


Figure 2. A permutation

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